Form and Structure Factors: Modeling and Interactions

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Outline

• Model fitting and least-squares methods
• Available form factors
  ex: sphere, ellipsoid, cylinder, spherical subunits…
  ex: polymer chain
• Monte Carlo integration for
  form factors of complex structures
• Monte Carlo simulations for
  form factors of polymer models
• Concentration effects and structure factors
  Zimm approach
  Spherical particles
  Elongated particles (approximations)
  Polymers
Motivation for ‘modelling’

- *not* to replace shape reconstruction and crystal-structure based modeling – we use the methods extensively

- alternative approaches to reduce the number of degrees of freedom in SAS data structural analysis (might make you aware of the limited information content of your data !!!)

- provide polymer-theory based modeling of flexible chains

- describe and correct for concentration effects
Literature


Jan Skov Pedersen
*Monte Carlo Simulation Techniques Applied in the Analysis of Small-Angle Scattering Data from Colloids and Polymer Systems*

in *Neutrons, X-Rays and Light*
P. Lindner and Th. Zemb (Editors) 2002 Elsevier Science B.V.
p. 381

Jan Skov Pedersen
*Modelling of Small-Angle Scattering Data from Colloids and Polymer Systems*

in *Neutrons, X-Rays and Light*
P. Lindner and Th. Zemb (Editors) 2002 Elsevier Science B.V.
p. 391

Rudolf Klein
*Interacting Colloidal Suspensions*

in *Neutrons, X-Rays and Light*
P. Lindner and Th. Zemb (Editors) 2002 Elsevier Science B.V.
p. 351
Form factors and structure factors

Warning 1:
Scattering theory – lots of equations!
= mathematics, Fourier transformations

Warning 2:
Structure factors:
Particle interactions = statistical mechanics

Not all details given
- but hope to give you an impression!
I will outline some calculations to show that it is not black magic!
Input data: Azimuthally averaged data

\[ q_i, I(q_i), \sigma[I(q_i)] \quad i = 1,2,3,...N \]

\( q_i \) calibrated

\( I(q_i) \) calibrated, i.e. on absolute scale
- noisy, (smeared), truncated

\( \sigma[I(q_i)] \) Statistical standard errors: Calculated from counting statistics by error propagation
- do not contain information on systematic error !!!!
Least-squared methods

Measured data: $I_{\text{exp}}(q_i), i = 1,\ldots,N$

Model: $I_{\text{mod}}(q_i), \alpha_i, i = 1,\ldots,M.$

Chi-square: 
\[ \chi^2 = \sum_{i=1}^{N} \left( \frac{I_{\text{exp}}(q_i) - I_{\text{mod}}(q_i)}{\sigma_i} \right)^2 \]

Reduced Chi-squared: 
\[ \chi_r^2 = \frac{\chi^2}{N - M} = \text{goodness of fit (GoF)} \]

Note that $\chi_r^2 = 1$ for $N \gg M$ corresponds to 
\[ |I_{\text{exp}}(q_i) - I_{\text{mod}}(q_i)| = \sigma_i \]

i.e. statistical agreement between model and data
Cross section

\[ \frac{d\sigma(q)}{d\Omega} : \text{number of scattered neutrons or photons per unit time, relative to the incident flux of neutron or photons, per unit solid angle at } q \text{ per unit volume of the sample.} \]

For system of **monodisperse** particles

\[ \frac{d\sigma(q)}{d\Omega} = I(q) = n \Delta\rho^2 V^2 P(q) S(q) \]

\[ = c M \Delta\rho_m^2 P(q) S(q) \]

\[ n \text{ is the number density of particles,} \]
\[ \Delta\rho \text{ is the excess scattering length density,} \]
\[ \text{given by electron density differences} \]
\[ V \text{ is the volume of the particles,} \]
\[ P(q) \text{ is the particle form factor; } P(q=0)=1 \]
\[ S(q) \text{ is the particle structure factor; } S(q=\infty)=1 \]

- \( V \propto M \)
- \( n = c/M \)
- \( \Delta\rho \) can be calculated from partial specific density, composition
Form factors of geometrical objects
Form factors I

Homogenous rigid particles

1. Homogeneous sphere
2. Spherical shell:
3. Spherical concentric shells:
4. Particles consisting of spherical subunits:
5. Ellipsoid of revolution:
   6. Tri-axial ellipsoid:
7. Cube and rectangular parallelepipeds:
8. Truncated octahedra:
9. Faceted Sphere:
   9x Lens
10. Cube with terraces:
11. Cylinder:
12. Cylinder with elliptical cross section:
13. Cylinder with hemi-spherical end-caps:
13x Cylinder with ‘half lens’ end caps
14. Toroid:
15. Infinitely thin rod:
16. Infinitely thin circular disk:
17. Fractal aggregates:
Form factors II

18. Flexible polymers with Gaussian statistics:
19. Polydisperse flexible polymers with Gaussian statistics:
20. Flexible ring polymers with Gaussian statistics:
21. Flexible self-avoiding polymers:
22. Polydisperse flexible self-avoiding polymers:
23. Semi-flexible polymers without self-avoidance:
24. Semi-flexible polymers with self-avoidance:
24x Polyelectrolyte Semi-flexible polymers with self-avoidance:
25. Star polymer with Gaussian statistics:
26. Polydisperse star polymer with Gaussian statistics:
27. Regular star-burst polymer (dendrimer) with Gaussian statistics:
28. Polycondensates of $A_f$ monomers:
29. Polycondensates of $AB_f$ monomers:
30. Polycondensates of $ABC$ monomers:
31. Regular comb polymer with Gaussian statistics:
32. Arbitrarily branched polymers with Gaussian statistics:
33. Arbitrarily branched semi-flexible polymers:
34. Arbitrarily branched self-avoiding polymers: (Block copolymer micelle)
35. Sphere with Gaussian chains attached:
36. Ellipsoid with Gaussian chains attached:
37. Cylinder with Gaussian chains attached:
38. Polydisperse thin cylinder with polydisperse Gaussian chains attached to the ends:
Form factors III

\[ P(q) = P_{\text{cross-section}}(q) P_{\text{large}}(q) \]

40. Very anisotropic particles with local planar geometry:
Cross section:
(a) Homogeneous cross section
(b) Two infinitely thin planes
(c) A layered centro symmetric cross-section
(d) Gaussian chains attached to the surface
Overall shape:
(a) Infinitely thin spherical shell
(b) Elliptical shell
(c) Cylindrical shell
(d) Infinitely thin disk

41. Very anisotropic particles with local cylindrical geometry:
Cross section:
(a) Homogeneous circular cross-section
(b) Concentric circular shells
(c) Elliptical Homogeneous cross section.
(d) Elliptical concentric shells
(e) Gaussian chains attached to the surface
Overall shape:
(a) Infinitely thin rod
(b) Semi-flexible polymer chain with or without excluded volume
From factor of a solid sphere

\[ A(q) = 4\pi \int_0^\infty \rho(r) \frac{\sin(qr)}{qr} r^2 dr = 4\pi \int_0^R \frac{\sin(qr)}{qr} r^2 dr \]

\[ = \frac{4\pi}{q} \int_0^R \sin(qr) rdr = \]

(partial integration)…

\[ = \frac{4\pi}{q} \left( - \frac{R \cos qR}{q} + \left[ \frac{\sin qr}{q} \right]_0^R \right) \]
\[ = \frac{4\pi}{q} \left( - \frac{R \cos qR}{q} + \frac{\sin qr}{q^2} \right) \]
\[ = \frac{4\pi}{q^3} (\sin qR - qR \cos qR) \]
\[ = \frac{4}{3} \pi R^3 \frac{3[\sin(qR) - qR \cos(qR)]}{(qR)^3} \text{ spherical Bessel function} \]
Form factor of sphere

\[ P(q) = A(q)^2 / V^2 \]
Measured data from solid sphere (SANS)

\[ I(\langle q \rangle) = \int R(\langle q \rangle, q) \frac{d\sigma(q)}{d\Omega} \, dq \]

\[ \frac{d\sigma}{d\Omega}(q) = \Delta \rho^2 V^2 \left[ \frac{3[\sin(qR) - qR \cos(qR)]}{(qR)^3} \right]^2 \]

Instrumental smearing is routinely included in SANS data analysis

Data from Wiggnal et al.
Ellipsoid

Prolates \((R, R, \varepsilon R)\)
\(\varepsilon > 1\)

Oblates \((R, R, \varepsilon R)\)
\(\varepsilon < 1\)

\[
P(q) = \int_0^{\pi/2} \left[ \Phi(qR') \right]^2 \sin \alpha \, d\alpha
\]

\[
R' = R(\sin^2 \alpha + \varepsilon^2 \cos^2 \alpha)^{1/2}
\]

\[
\Phi(x) = \frac{3[\sin(x) - x \cos(x)]}{x^3}
\]
P(q): Ellipsoid of revolution

- Prolate ellipsoid of revolution 1:1:3 (-----)
- Oblate ellipsoid of revolution 1:1:0.2 (-----)
Lysozyme 7 mg/mL

Ellipsoid of revolution + background

$R = 15.48 \, \text{Å}$

$\varepsilon = 1.61$ (prolate)

$\chi^2 = 2.4$ ($\chi = 1.55$)
Core-shell particles:

\[ A_{\text{core-shell}}(q) = \left[ \Delta \rho_{\text{shell}} V_{\text{out}} \Phi(q R_{\text{out}}) - (\Delta \rho_{\text{shell}} - \Delta \rho_{\text{core}}) V_{\text{in}} \Phi(q R_{\text{in}}) \right] \]

where

\[ V_{\text{out}} = 4\pi R_{\text{out}}^3/3 \quad \text{and} \quad V_{\text{in}} = 4\pi R_{\text{in}}^3/3. \]

\( \Delta \rho_{\text{core}} \) is the excess scattering length density of the core, 
\( \Delta \rho_{\text{shell}} \) is the excess scattering length density of the shell and:

\[ \Phi(x) = \frac{3[\sin x - x \cos x]}{x^3} \]
$R_{\text{out}} = 30 \ \text{Å}$

$R_{\text{core}} = 15 \ \text{Å}$

$\Delta \rho_{\text{shell}} = 1$

$\Delta \rho_{\text{core}} = -1$. 
SDS micelle

Hydrocarbon core

Headgroup/counterions

20 Å
SDS micelles:
prolate ellipsoid with shell of constant thickness

\[ \chi^2 = 2.3 \]

- \[ I(0) = 0.0323 \pm 0.0005 \text{ 1/cm} \]
- \[ R_{\text{core}} = 13.5 \pm 2.6 \text{ Å} \]
- \[ \varepsilon = 1.9 \pm 0.10 \]
- \[ d_{\text{head}} = 7.1 \pm 4.4 \text{ Å} \]
- \[ \rho_{\text{head}}/\rho_{\text{core}} = -1.7 \pm 1.5 \]
- \[ \text{backgr} = 0.00045 \pm 0.00010 \text{ 1/cm} \]
Molecular constraints:

\[
\Delta \rho_{\text{tail}}^e = \frac{Z_{\text{tail}}^e}{V_{\text{tail}}} - \frac{Z_{\text{H}_2\text{O}}^e}{V_{\text{H}_2\text{O}}}, \quad V_{\text{core}} = N_{\text{agg}} V_{\text{tail}}
\]

\[
\Delta \rho_{\text{head}}^e = \frac{Z_{\text{head}}^e + nZ_{\text{H}_2\text{O}}}{V_{\text{head}} + nV_{\text{H}_2\text{O}}} - \frac{Z_{\text{H}_2\text{O}}^e}{V_{\text{H}_2\text{O}}}
\]

\[
n_{\text{micelles}} = \frac{c}{N_{\text{agg}} M_{\text{surfactant}}}
\]

\[n\text{ water molecules in headgroup shell}\]

\[
V_{\text{shell}} = N_{\text{agg}} (V_{\text{head}} + nV_{\text{H}_2\text{O}})
\]
Cylinder

\[ P(q) = \int_0^{\pi/2} \left[ \frac{2J_1(qR \sin \alpha)}{qR \sin \alpha} \frac{\sin(qL \cos \alpha / 2)}{qL \cos \alpha / 2} \right]^2 \sin \alpha \, d\alpha \]

\( J_1(x) \) is the Bessel function of first order and first kind.
\[ L \gg R \quad P(q) \Rightarrow P_{\text{cross-section}}(q) P_{\text{large}}(q) \]
Glucagon Fibrils

\[ R = 29 \text{Å} \]

\[ R = 16 \text{Å} \]

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Cristiano Luis Pinto Oliveira, Manja A. Behrens, Jesper Søndergaard Pedersen, Kurt Erlacher, Daniel Otzen and Jan Skov Pedersen  
Primus

Modeling with geometrical bodies

Type of body:
1. ellipsoid
2. ellipsoid of rotation
3. cylinder
4. elliptical cylinder
5. hollow cylinder
6. rectangular prism
7. hollow sphere
8. all types

Experimental data
Select

Angular units

Nbeg
1
Nend
9999

Apply
Clear
Cancel

Primus Information window

Konarev P.V., Volkov V.V., Sokolova A.V.,
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| 2003, | V. 36, | p. 1277-1282 |

Running
Input pending in Graphics Window
Form factor for particles with arbitrary shape

Spherical monodisperse particles

\[ I(q) = P_{sphere}(q, R) \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\sin(qr_{ij})}{qr_{ij}} \]
Monte Carlo integration in calculation of form factors for complex structures

- Generate points in space by Monte Carlo simulations
- Select subsets by geometric constraints
- Calculate histograms $p(r)$ functions
- (Include polydispersity)
- Fourier transform

$\sum x(i)^2 + y(i)^2 + z(i)^2 \leq R^2$
Protein

Micelle

ISCOM vaccine particle
Immune-stimulating complexes: ISCOMs
- Self-assembled vaccine nano-particles

Stained TEM

Quil-A

ISCOMs typically:
60–70wt% Quil-A,
10–15wt% Phospholipids
10-15wt% cholesterol

M T Sanders et al.
*Immunology and Cell Biology (2005)* 83, 119–128
SAXS data

Investigation of the structure of ISCOM particles by SAXS

Oscillations ⇒ relatively monodisperse
Bump at high q ⇒ core-headgroup structure
Combination of icosahedrons, tennis balls and footballs

Fit result:
Icosah: $\varphi_m \ M = 0.057$
Tennis: $\varphi_m \ M = 0.770$
Football: $\varphi_m \ M = 0.173$

From volumes of cores:
Icosah: $M = 0.557$ a.u.
Tennis: $M = 1.000$ a.u.
Football: $M = 1.616$ a.u.

Mass distribution:
Icosah: $\varphi_m = 0.104$
Tennis: $\varphi_m = 0.786$
Football: $\varphi_m = 0.110$
Icosahedrons, tennis ball, football

Hydrophobic core: (2 x) 11 Å
Hydrophilic headgroup region: Width 13 Å
Pure SDS micelles in buffer:

C > CMC (~ 5 mM)

N_{agg} = 66±1 oblate ellipsoids
R=20.3±0.3 Å
ε=0.663±0.005

C ≤ CMC
Polymer chains in solution

Gigantic ensemble of 3D random flights
- all with different configurations
Gaussian polymer chain

Look at two points: Contour separation: $L$
Spatial separation: $r$

Contribution to scattering: $I_2(q) = \frac{\sin(qr)}{qr}$

For an ensemble of polymers, points with $L$ has $<r^2> = Lb$
and $r$ has a Gaussian distribution: $D(r) \propto \exp\left[-\frac{3r^2}{2 <r^2>}\right]$

Add scattering from all pair of points

‘Density of points’: $(L_o - L)$
Gaussian chains: The calculation

\[
P(q) = \frac{1}{L^2} \int_0^\infty dr \int_0^{L_o} dL_1 \int_0^{L_o} dL_2 D(r, |L_2 - L_1|) \frac{\sin(qr)}{qr} r
\]

\[
= \frac{1}{L_o} \int_0^\infty dr \int_0^{L_o} dL(L_o - L) D(r, L) \frac{\sin(qr)}{qr} r^2
\]

\[
= \frac{1}{L_o} \int_0^{L_o} dL(L_o - L) \exp\left[-q^2 Lb / 6\right]
\]

\[
= \frac{2[\exp(-x) - 1 + x]}{x^2} \quad x = R_g^2 q^2 \quad R_g^2 = Lb/6
\]

How does this function look?
Polymer scattering

Form factor of Gaussian chain

\[ q \approx \frac{1}{R_g} \]

\[ q^{-\frac{1}{\nu}} = q^{-2} \]
Gaussian chain + background
\[ R_g = 21.3 \ \text{Å} \]
\[ \chi^2 = 1.8 \]

\[ \Delta \rho \text{ is low!} \]
Self avoidance

No excluded volume

\[ P(x = R_g^2 q^2) = \frac{2[\exp(-x) - 1 + x]}{x^2} \]

No excluded volume

\[ \Rightarrow \text{expansion} \]

no analytical solution!
Monte Carlo simulation approach

(1) Choose model

(2) Vary parameters in a broad range
   Generate configs., sample \( P(q) \)

(3) Analyze \( P(q) \) using physical insight

(4) Parameterize \( P(q) \) using physical insight

(5) Fit experimental data using numerical expressions for \( P(q) \)

\[
P(q, L, b)
\]

\( L = \) contour length

\( b = \) Kuhn (‘step’) length

Pedersen and Schurtenberger 1996
Expansion = 1.5

Form factor polymer chains

\[ q \approx \frac{1}{R_g} (Gaussian) \]

\[ q \approx \frac{1}{R_g} (excl. vol.) \]

Excluded volume chains:

\[ q^{-\frac{1}{\nu}} = q^{-1.70} \]

\[ \nu = 0.588 \]

Gaussian chains

\[ q^{-\frac{1}{\nu}} = q^{-2} \]

\[ \nu = 1/2 \]

local stiffness
C16E6 micelles with ‘C

Fig. 3. Fit to the SANS data from doped C16E6 micelles (doping level of 6% of ionic surfactant) with different salt concentrations at low surfactant concentrations (c = 0.4 mg/ml) [32*]. From top to bottom: 0.1, 0.01, 0.005, 0.0025 and 0.001 M salt.
Models II

Polydispersity: Spherical particles as e.g. vesicles

No interaction effects: Size distribution $D(R)$

$$\frac{d\sigma(q)}{d\Omega} = \Delta \rho^2 \int_0^\infty D(R) V(R)^2 P(q, R) \, dR.$$  

Oligomeric mixture: Discrete particles

$$\frac{d\sigma(q)}{d\Omega} = c \Delta \rho^2 \sum_i M_i f_i P_i(q)$$

Application to insulin:


$f_i = \text{mass fraction} \quad \sum_i f_i = 1$

Used in PRIMUS ‘Oligomers’
Concentration effects
Concentration effects in protein solutions

**α-crystallin**
eye lens protein

Fig. 1. Variation of normalized scattered intensity of α-crystallin solutions as a function of protein concentration, at 20°C, in 150 mM phosphate buffer at pH = 6.8.

S. Finet*1, A. Tardieu
p(r) by Indirect Fourier Transformation (IFT)

- as you have done by GNOM!

At high concentration, the neighborhood is different from the average further away!

1. Simple approach: Exclude low $q$ data.
2. Glatter: Use Generalized Indirect Fourier Transformation (GIFT)

I. Pilz, 1982

At high concentration, the neighborhood is different from the average further away!
Low concentrations  Zimm 1948 – originally for light scattering

\[ P'(q) = P(q) - \nu P(q)^2 \]
\[ = P(q)[1 - \nu P(q)] \]

\( \nu \sim \text{concentration} \)

Higher order terms:
\[ P'(q) = P(q)[1 - \nu P(q) \{1 - \nu P(q)\}] \]
\[ = P(q)[1 - \nu P(q) + \nu^2 P(q)^2] \]

\[ = P(q)[1 - \nu P(q) + \nu^2 P(q)^2 - \nu^3 P(q)^3 \ldots\]}
\[ = \frac{P(q)}{1 + \nu P(q)} \]
\[ = \text{Random-Phase Approximation (RPA)} \]
**Zimm approach**

\[ I(q) = K \frac{P(q)}{1 + \nu P(q)} \]

\[ I(q)^{-1} = K^{-1} \frac{1 + \nu P(q)}{P(q)} = K^{-1} \left[ \frac{1}{P(q)} + \nu \right] \]

With \( P(q) \approx \frac{1}{1 + q^2 R_g^2 / 3} \)

\[ I(q)^{-1} = K^{-1} \left[ 1 + q^2 R_g^2 / 3 + \nu \right] \]

Plot \( I(q)^{-1} \) versus \( q^2 + c \) and extrapolate to \( q=0 \) and \( c=0 \)!
Zimm plot

Kirste and Wunderlich
PS in toluene

\[ I(q)^{-1} \]

\[ P(q) \]

\[ q = 0 \]

\[ q^2 + c \]
My suggestion:

( - which includes also information from what follows)

• Minimum 3 concentrations for same system.
• Fit data simultaneously all data sets

\[
\frac{I(q_i)}{c} = \frac{P_i}{1 + ca_1 \exp(-q_i^2 a_2^2 / 3)}
\]

with \( a_1, a_2, \) and \( P_i \) as fit parameters
Osteopontin (M=35 kDa) in water and 10 mM NaCl

2.5, 5 and 10 mg/mL

Shipovskov, Oliveira, Hoffmann, Schauser, Sutherland, Besenbacher, Pedersen (2012)
Shipovskov, Oliveira, Hoffmann, Schauer, Sutherland, Besenbacher, Pedersen (2012)
Shipovskov, Oliveira, Hoffmann, Schauer, Sutherland, Besenbacher, Pedersen (2012)
But now we look at the information content related to these effects…

Understand the fundamental processes and principles governing aggregation and crystallization

Why is the eye lens transparent despite a protein concentration of 30-40%?
**Structure factor**

Spherical monodisperse particles

\[ I(q) = V^2 \Delta \rho^2 P(q) \sum \frac{\sin qr_{jk}}{qr_{jk}} \]

\[ = V^2 \Delta \rho^2 P(q) S(q) \]

\[ = V^2 \Delta \rho^2 P(q) \sum p(r_j) \frac{\sin qr_j}{qr_j} \]

\( S(q) \) is related to the probability distribution function of inter-particles distances, i.e. the pair correlation function \( g(r) \)
Correlation function $g(r)$

$g(r) =$ \textit{Average} of the normalized density of atoms in a shell $[r; r+dr]$ from the center of a particle
\( g(r) \) and \( S(q) \)

\[
S(q) = 1 + n \frac{4\pi}{q} \int (g(r) - 1) \frac{\sin(qr)}{qr} r^2 \, dr
\]
Glatter: Generalized Indirect Fourier Transformation (GIFT)

\[ I(q) = 4\pi \int p(r) \frac{\sin(qr)}{qr} \, dr \]

With concentration effects

\[ I(q) = S_{\text{eff}}(q, \eta, R, \sigma(R), Z, c_{\text{salt}}) 4\pi \int p(r) \frac{\sin(qr)}{qr} \, dr \]

Optimized by constrained non-linear least-squares method

- works well for globular models and provides \( p(r) \)
A SAXS study of the small hormone glucagon: equilibrium aggregation and fibrillation

Home-written software

A SAXS Study of Glucagon Fibrillation
Cristiano Luis Pinto Oliveira\textsuperscript{1*}, Manja Annette Behrens\textsuperscript{1}, Jesper Søndergaard Pedersen\textsuperscript{2}, Kurt Erlacher\textsuperscript{1}, Daniel Otzen\textsuperscript{2,3} and Jan Skov Pedersen\textsuperscript{1}

Osmometry (Second virial coeff $A_2$)

$$\Pi = c \left( \frac{RT}{M} \right) \left( 1 + A_2 c + A_3 c^2 + \ldots \right)$$
$S(q)$, virial expansion and Zimm

From statistical mechanics…:

$$S(q = 0) = RT \left( \frac{\partial \Pi}{\partial c} \right)^{-1} = \frac{1}{1 + 2cMA_2 + 3c^2MA_3 + ...}$$

⇒ In Zimm approach $\nu = 2cMA_2$
A$_2$ in lysozyme solutions

O. D. Velev, E. W. Kaler, and A. M. Lenhoff

FIGURE 2 A summary of virial coefficients of lysozyme obtained from the SLS data at varying pH and four electrolyte concentrations (each point corresponds to one data set similar to those plotted in Fig. 1). The open symbols denote experimental conditions where the onset of aggregation was detected.
Colloidal interactions

- Excluded volume ‘repulsive’ interactions (‘hard-sphere’)
- Short range attractive van der Waals interaction (‘stickiness’)
- Short range attractive hydrophobic interactions (solvent mediated ‘stickiness’)
- Electrostatic repulsive interaction (or attractive for patchy charge distribution!)
  (effective Debye-Hückel potential)
- Attractive depletion interactions (co-solute (polymer) mediated)

\[
v_c(r) = \frac{Q^2}{\varepsilon} \frac{e^{-\kappa(r-\sigma)}}{\left(1 + \frac{\kappa \sigma}{2}\right)^2} \frac{1}{r}
\]
Theory for colloidal stability

DLVO theory: (Derjaguin-Landau-Vewey-Overbeek)

Debye-Hückel
screened Coulomb potential
+ attractive interaction

\[ v_c(r) = \frac{Q^2}{\varepsilon \left(1 + \frac{\kappa \sigma}{2}\right)^2} \frac{e^{-\kappa(r-\sigma)}}{r} \]
Depletion interactions

\[ v_{\text{eff}}(r; \rho_1, \rho_2, T) = \begin{cases} \infty; & r < \sigma_1 \\ v_{A0}(r); & \sigma_1 < r < \sigma_1 + \sigma_2 \\ 0; & r > \sigma_1 + \sigma_2 \end{cases} \]

where

\[ v_{A0}(r) = -k_B T \rho_2 \frac{\pi \sigma_1^3(1+q)^3}{6} \left[ 1 - \frac{3r}{2(1+q)\sigma_1} + \frac{r^3}{2(1+q)^3\sigma_1^3} \right] \]

with \( q = \sigma_2/\sigma_1 \) the ratio of diameters.

Asakura & Oosawa, 58
Integral equation theory

Relate $g(r)$ [or $S(q)$] to $V(r)$

At low concentration $g(r) = \exp\left(- \frac{V(r)}{k_B T}\right) \approx 1 - \frac{V(r)}{k_B T}$

Boltzmann approximation (weak interactions)

Make expansion around uniform state [Ornstein-Zernike eq.]

$g(r) = 1 - nc(r) - [3 \text{ particle}] - [4 \text{ particle}] - \ldots$

$= 1 - nc(r) - n^2 c(r) \ast c(r) - n^3 c(r) \ast c(r) \ast c(r) - \ldots$

[* = convolution] $\quad c(r) = \text{direct correlation function}$

$S(q) = 1 - nc(q) - n^2 c(q)^2 - n^3 c(q)^3 - \ldots = \frac{1}{1 - nc(q)}$

$\Rightarrow$ but we still need to relate $c(r)$ to $V(r)$ !!!
Closure relations

Systematic density expansion….

Mean-spherical approximation MSA: $c(r) = -\frac{V(r)}{k_BT}$
(analytical solution for screened Coulomb potential
  - but not accurate for low densities)

Percus-Yevick approximation PY: $c(r) = g(r) \left[ \exp\left(\frac{V(r)}{k_BT}\right) - 1 \right]$
(analytical solution for hard-sphere potential + sticky HS)

Hypernetted chain approximation HNC:
  $c(r) = -\frac{V(r)}{k_BT} + g(r) - 1 - \ln(g(r))$
(Only numerical solution
  - but quite accurate for Coulomb potential)

Rogers and Young closure RY:
  Combines PY and HNC in a self-consistent way
(Only numerical solution
  - but very accurate for Coulomb potential)
**α-crystallin**

S. Finet, A. Tardieu

**DLVO potential**

\[
v(r) = \begin{cases} 
\frac{Z_p^2 e^2}{4\pi \varepsilon_0 \varepsilon k Tr(1 + 0.5\sigma/\lambda_D)^2} & \text{for } r > \sigma, \\
\times \exp \left[-\frac{(r - \sigma)}{\lambda_d}\right] & \\
-J\sigma/r \exp \left[-\frac{(r - \sigma)}{d}\right] & \\
+\infty & \text{for } r \leq \sigma.
\end{cases}
\]

**HNC and numerical solution**

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Fig. 1. Variation of normalized scattered intensity of α-crystallin solutions as a function of protein concentration, at 20°C, in 150 mM phosphate buffer at pH = 6.8.
**$\alpha$-crystallin + PEG 8000**  
S. Finet, A. Tardieu

**Depletion interactions:**

**DLVO potential**

\[
v(r) = \begin{cases} 
\frac{Z_p^2 e^2}{4\pi \varepsilon \varepsilon_0 k T (1 + 0.5 \sigma / \lambda_D)^2} \times \exp \left[ -\frac{(r - \sigma)}{\lambda_d} \right] \\
-J \sigma / r \exp \left[ -(r - \sigma)/d \right] + \infty
\end{cases}
\]

for $r > \sigma$,  
for $r \leq \sigma$.

**HNC and numerical solution**

40 mg/ml $\alpha$-crystallin solution pH 6.8,  
150 mM ionic strength.
Anisotropy

\[
\frac{d\sigma(q)}{d\Omega} = \Delta \rho^2 V^2 \left[ \sum_i F_i(q,e_i)^2 + \frac{1}{N} \sum_{i,j} F_i(q,e_i) F_j(q,e_j) [S_{i,j}(q,e_i,e_j) - 1] \right]
\]

Small: decoupling approximation (Kotlarchyk and Chen, 1984):

\[
\frac{d\sigma(q)}{d\Omega} = \Delta \rho^2 V^2 P(q) [1 + \beta(q)(S(q) - 1)],
\]

\[
\beta(q) = \langle F(q) \rangle^2 / \langle F^2(q) \rangle_o,
\]

Measured structure factor:

\[
S_{\text{meas}}(q) \equiv \frac{\partial \sigma(q)}{n \Delta \rho^2 P(q)} = 1 + \beta(q) [S(q) - 1] \neq S(q) \quad !!!!!
\]
Large anisotropy

Polymers, cylinders…

Anisotropy, large: Random-Phase Approximation (RPA):

\[
\frac{d\sigma}{d\Omega}(q) = n\Delta \rho^2 V^2 \frac{P(q)}{1 + \nu P(q)} \quad \nu \sim \text{concentration}
\]

Anisotropy, large: Polymer Reference Interaction Site Model (PRISM)
Integral equation theory – equivalent site approximation

\[
\frac{d\sigma}{d\Omega}(q) = n\Delta \rho^2 V^2 \frac{P(q)}{1 - n c(q) P(q)} \quad c(q) \text{ direct correlation function related to FT of } V(r)
\]
Empirical PRISM expression

\[ \frac{d\sigma}{d\Omega}(q) = n\Delta \rho^2 V^2 \frac{P(q)}{1 - nc(q)P(q)} \]

\(c(q) = \text{rod formfactor}
- \text{empirical from MC simulation}\)

SDS micelle in 1 M NaBr

Arleth, Bergström and Pedersen
Overview: Available Structure factors

1) **Hard-sphere potential**: Percus-Yevick approximation
2) **Sticky hard-sphere potential**: Percus-Yevick approximation
3) **Screened Coulomb potential**:
   - Mean-Spherical Approximation (MSA).
   - Rescaled MSA (RMSA).
   - Thermodynamically self-consistent approaches (Rogers and Young closure)
4) **Hard-sphere potential, polydisperse system**: Percus-Yevick approximation
5) **Sticky hard-sphere potential, polydisperse system**: Percus-Yevick approximation
6) **Screened Coulomb potential, polydisperse system**: MSA, RMSA,
7) **Cylinders, RPA**
8) **Cylinders, `PRISM'**:
9) **Solutions of flexible polymers, RPA**: 
10) **Solutions of semi-flexible polymers, `PRISM'**: 
11) **Solutions of polyelectrolyte chains 'PRISM'**: 
Summary

• Model fitting and least-squares methods
• Available form factors
  ex: sphere, ellipsoid, cylinder, spherical subunits…
  ex: polymer chain
• Monte Carlo integration for
  form factors of complex structures
• Monte Carlo simulations for
• form factors of polymer models

• Concentration effects and structure factors
  Zimm approach
  Spherical particles
  Elongated particles (approximations)
  Polymers
Literature


Jan Skov Pedersen
*Monte Carlo Simulation Techniques Applied in the Analysis of Small-Angle Scattering Data from Colloids and Polymer Systems*
in *Neutrons, X-Rays and Light*
P. Lindner and Th. Zemb (Editors) 2002 Elsevier Science B.V.
p. 381

Jan Skov Pedersen
*Modelling of Small-Angle Scattering Data from Colloids and Polymer Systems*
in *Neutrons, X-Rays and Light*
P. Lindner and Th. Zemb (Editors) 2002 Elsevier Science B.V.
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Rudolf Klein
*Interacting Colloidal Suspensions*
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p. 351